Existence of positive solutions for a critical nonlinear Schrödinger equation with vanishing or coercive potentials

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Abstract: In this paper we investigate the existence of the positive solutions for the following nonlinear Schrödinger equation

$$-\triangle u+V(x)u=K(x)|u|^{p-2}u \text{ in } \mathbb{R}^N$$

where $V(x)\sim a|x|^{-b}$ and $K(x)\sim \mu|x|^{-s}$ as $|x|\to\infty$ with $0<a,\mu<+\infty$, $b<2$, $b\neq 0$, $0<\frac{s}{b}<1$ and $p=2(N-2s/b)/(N-2)$.

Key words: semilinear Schrödinger equation, vanishing or coercive potentials.
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1 Introduction and statement of results

In this paper, we consider the following semilinear elliptic equation

$$-\triangle u+V(x)u=K(x)|u|^{p-2}u \text{ in } \mathbb{R}^N. \tag{1.1}$$

where $N\geq 3$. The exponent

$$p=2(N-\frac{2s}{b})/(N-2) \tag{1.2}$$

with the real numbers b and s satisfying

$$b<2,\ b\neq 0,\ 0<\frac{s}{b}<1. \tag{1.3}$$

By this definition, $2<p<2^*:=2N/(N-2)$.

With respect to the functions V and K, we assume

$(\mathbf{A_1})$. $V,K\in C(\mathbb{R}^N)$. For every $x\in\mathbb{R}^N$, $V(x)>0$ and $K(x)>0$.

$(\mathbf{A_2})$. There exist $0<a<\infty$ and $0<\mu<\infty$ such that

$$\lim_{|x|\to\infty}|x|^bV(x)=a \text{ and } \lim_{|x|\to\infty}|x|^sK(x)=\mu. \tag{1.4}$$

A typical example for Eq. (1.1) with V and K satisfying $(\mathbf{A_1})$ and $(\mathbf{A_2})$ is the equation

$$-\triangle u+\frac{a}{(1+|x|)^b}u=\frac{\mu}{(1+|x|)^s}|u|^{p-2}u \text{ in } \mathbb{R}^N \tag{1.5}$$

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When $0 < b < 2$, the potentials are vanishing at infinity and when $b < 0$, the potentials are coercive.

Eq.(1.1) arises in various applications, such as chemotaxis, population genetics, chemical reactor theory, and the study of standing wave solutions of certain nonlinear Schrödinger equations. Therefore, they have received growing attention in recent years (one can see, e.g., [2], [3], [5], [10], [11] and [13] for reference).

Under the above assumptions, Eq.(1.1) has a natural variational structure. For an open subset Ω in $\mathbb{R}^N$, let $C_0^\infty(\Omega)$ be the collection of smooth functions with compact support set in Ω. Let E be the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the inner product

$$(u,v)_E = \int_{\mathbb{R}^N} \nabla u \nabla v dx + \int_{\mathbb{R}^N} V(x) uv dx.$$

From the assumptions $(\mathbf{A_1})$ and $(\mathbf{A_2})$, we deduce that

$$(\int_{\mathbb{R}^N} \frac{|u|^2}{(1+|x|)^b} dx)^{1/2} \text{ and } (\int_{\mathbb{R}^N} V(x)|u|^2 dx)^{1/2}$$

are two equivalent norms in the space

$$L_V^2(\mathbb{R}^N) = \{u \text{ is measurable in } \mathbb{R}^N \mid \int_{\mathbb{R}^N} V(x)|u|^2 dx < +\infty\}.$$

Therefore, there exists $B_1 > 0$ such that

$$(\int_{\mathbb{R}^N} \frac{|u|^2}{(1+|x|)^b} dx)^{1/2} \le B_1 (\int_{\mathbb{R}^N} V(x)|u|^2 dx)^{1/2}.$$

Moreover, the assumptions $(\mathbf{A_1})$ and $(\mathbf{A_2})$ imply that there exists $B_2 > 0$ such that

$$K(x) \le B_2 (1+|x|)^{-s}, \ \forall x \in \mathbb{R}^N.$$

Then by the Hölder and the Sobolev inequalities (see, e.g., [14, Theorem 1.8]), we have, for every $u \in C_0^\infty(\mathbb{R}^N)$,

$$\begin{aligned}
(\int_{\mathbb{R}^N} K(x)|u|^p dx)^{\frac{1}{p}} &\le C(\int_{\mathbb{R}^N} \frac{|u|^p}{(1+|x|)^s} dx)^{\frac{1}{p}} \\
&= C(\int_{\mathbb{R}^N} \frac{|u|^{\frac{2s}{b}}}{(1+|x|)^s} \cdot |u|^{p-\frac{2s}{b}} dx)^{\frac{1}{p}} \\
&\le C(\int_{\mathbb{R}^N} \frac{|u|^2}{(1+|x|)^b} dx)^{\frac{s}{pb}} (\int_{\mathbb{R}^N} |u|^{2^*} dx)^{\frac{1}{p}(1-\frac{s}{b})} \\
&\le C(\int_{\mathbb{R}^N} \frac{|u|^2}{(1+|x|)^b} dx)^{\frac{s}{pb}} (\int_{\mathbb{R}^N} |\nabla u|^2 dx)^{\frac{2^*}{2p}(1-\frac{s}{b})} \\
&= C(\int_{\mathbb{R}^N} \frac{|u|^2}{(1+|x|)^b} dx)^{\frac{1}{2}\cdot\frac{2s}{pb}} (\int_{\mathbb{R}^N} |\nabla u|^2 dx)^{\frac{1}{2}\cdot(1-\frac{2s}{pb})} \\
&\le C(\int_{\mathbb{R}^N} V(x)|u|^2 dx)^{\frac{1}{2}\cdot\frac{2s}{pb}} (\int_{\mathbb{R}^N} |\nabla u|^2 dx)^{\frac{1}{2}\cdot(1-\frac{2s}{pb})},
\end{aligned}$$

where $C > 0$ is a constant independent of u. It follows that there exists a constant $C' > 0$ such that

$$(\int_{\mathbb{R}^N} K(x)|u|^p dx)^{1/p} \le C'(\int_{\mathbb{R}^N} |\nabla u|^2 dx)^{1/2} + C'(\int_{\mathbb{R}^N} V(x)|u|^2 dx)^{1/2}.$$

This implies that E can be embedded continuously into the weighted L^p-space

$$L_K^p(\mathbb{R}^N) = \{u \text{ is measurable in } \mathbb{R}^N \mid \int_{\mathbb{R}^N} K(x)|u|^p dx < +\infty\}.$$

Then the functional

$$\Phi(u)=\frac{1}{2}||u||_E^2-\frac{1}{p}\int_{\mathbb{R}^N}K(x)|u|^pdx,\ u\in E$$

is well defined in E. And it is easy to check that Φ is a C^2 functional and the critical points of Φ are solutions of (1.1) in E.

In a recent paper [1], Alves and Souto proved that the space E can be embedded compactly into $L^p_K(\mathbb{R}^N)$ if $0<b<2$ and $2(N-2s/b)/(N-2)<p<2^*$ and Φ satisfies Palais-Smale condition consequently. Then by using the mountain pass theorem, they obtained a nontrivial solution for Eq.(1.1). Unfortunately, when $p=2(N-2s/b)/(N-2)$, the embedding of E into $L^p_K(\mathbb{R}^N)$ is not compact and Φ satisfies no longer Palais-Smale condition. Therefore, the "standard" variational methods fail in this case. From this point of view, $p=2(N-2s/b)/(N-2)$ should be seen as a kind of critical exponent for Eq.(1.1). If the potentials V and K are restricted to the class of radially symmetric functions, "compactness" of such a kind is regained and "standard" variational approaches work (see [11] and [13]). But this method does not seem to apply to the more general equation (1.1) where K and V are non-radially symmetric functions.

It is not easy to deal with Eq. (1.1) directly because there are no known approaches can be used directly to overcome the difficulty brought by the loss of compactness. However, in this paper, through an interesting transformation, we find an equivalent equation for Eq. (1.1) (see Eq. (2.9) in Section 2). This equation has the advantages that its Palais-Smale sequence can be characterized precisely through the concentration-compactness principle (see Theorem 5.1) and it possesses partial compactness (see Corollary 5.8). By means of these advantages, a positive solution for this equivalent equation and then a corresponding positive solution for Eq. (1.1) are obtained.

Before to state our main result, we need to give some definitions.

Let

$$V_*(x)=|x|^{\frac{2b}{2-b}}V(|x|^{\frac{b}{2-b}}x)+C_b|x|^{-2},\tag{1.6}$$

where

$$C_b=\frac{b}{4}(1-\frac{b}{4})(N-2)^2\tag{1.7}$$

and

$$K_*(x)=|x|^{\frac{2s}{2-b}}K(|x|^{\frac{b}{2-b}}x).\tag{1.8}$$

Let $H^1(\mathbb{R}^N)$ be the the Sobolev space endowed with the norm and the inner product

$$||u||=(\int_{\mathbb{R}^N}|\nabla u|^2dx+\int_{\mathbb{R}^N}u^2dx)^{1/2}\text{ and }(u,v)=\int_{\mathbb{R}^N}(\nabla u\cdot\nabla v+uv)dx$$

respectively and $L^p(\mathbb{R}^N)$ be the function space consisting of the functions on $\mathbb{R}^N$ that are $p-$integrable. Since $2<p<2^*$, $H^1(\mathbb{R}^N)$ can be embedded continuously into $L^p(\mathbb{R}^N)$. Therefore, the infimum

$$\inf_{v\in H^1(\mathbb{R}^N)\setminus\{0\}}\frac{\int_{\mathbb{R}^N}|\nabla v|^2dx+a\int_{\mathbb{R}^N}v^2dx}{(\int_{\mathbb{R}^N}|v|^pdx)^{2/p}}>0.\tag{1.9}$$

We denote this infimum by S_p.

Our main result reads as follows:

Theorem 1.1. *Under the assumptions* $(\mathbf{A_1})$ *and* $(\mathbf{A_2})$, *if* b, s *and* p *satisfy* (1.3) *and* (1.2) *and*

$$\begin{aligned}&\inf_{u\in H^1(\mathbb{R}^N)\setminus\{0\}}\frac{\int_{\mathbb{R}^N}|\nabla u|^2dx+(\frac{b^2}{4}-b)\int_{\mathbb{R}^N}\frac{|x\cdot\nabla u|^2}{|x|^2}dx+\int_{\mathbb{R}^N}V_*(x)|u|^2dx}{(\int_{\mathbb{R}^N}K_*(x)|u|^pdx)^{2/p}}\\ <\ &(1-b/2)^{\frac{p-2}{p}}\mu^{-\frac{2}{p}}S_p,\end{aligned}\tag{1.10}$$

then Eq. (1.1) *has a positive solution* $u\in E$.

Remark 1.2. *We should emphasize that the condition* (1.10) *can be satisfied in many situations. For* $r > 0$, *let* $R_r = \{x \in \mathbb{R}^N \mid r/2 < |x| < r\}$ *and* $H_0^1(R_r)$ *be the closure of* $C_0^\infty(R_r)$ *in* $H^1(\mathbb{R}^N)$. *Under the assumptions* $(\mathbf{A_1})$ *and* $(\mathbf{A_2})$, *we have*

$$\inf_{u \in H_0^1(R_r)\setminus\{0\}} \frac{\int_{R_r} |\nabla u|^2 dx}{(\int_{R_r} K_*(x)|u|^p dx)^{2/p}} \to 0, \text{ as } r \to +\infty.$$

Then for any $\epsilon > 0$, *there exist* $r_\epsilon > 0$ *and* $u_\epsilon \in H_0^1(R_r) \setminus \{0\}$ *such that*

$$\frac{\int_{R_r} |\nabla u_\epsilon|^2 dx}{(\int_{R_r} K_*(x)|u_\epsilon|^p dx)^{2/p}} < \epsilon.$$

It follows from this inequality and $\int_{R_r} \frac{|x\cdot\nabla u_\epsilon|^2}{|x|^2} dx \le \int_{R_r} |\nabla u_\epsilon|^2 dx$ *that if* $\sup_{R_r} V_*$ *is small enough such that*

$$\frac{\int_{R_r} V_*(x)|u_\epsilon|^2 dx}{(\int_{R_r} K_*(x)|u_\epsilon|^p dx)^{2/p}} < \epsilon,$$

then

$$\begin{aligned} &\frac{\int_{R_r} |\nabla u_\epsilon|^2 dx + (\frac{b^2}{4} - b)\int_{R_r} \frac{|x\cdot\nabla u_\epsilon|^2}{|x|^2} dx + \int_{R_r} V_*(x)|u_\epsilon|^2 dx}{(\int_{R_r} K_*(x)|u_\epsilon|^p dx)^{2/p}} \\ < \quad & (2 + |\frac{b^2}{4} - b|)\epsilon \end{aligned}$$

This implies that (1.10) *is satisfied if* ϵ *is chosen such that* $(2 + |\frac{b^2}{4} - b|)\epsilon < (1 - b/2)^{\frac{p-2}{p}} \mu^{-\frac{2}{p}} S_p$.

Notations: Let X be a Banach Space and $\varphi \in C^1(X, \mathbb{R})$. We denote the Fréchet derivative of φ at u by $\varphi'(u)$. The Gateaux derivative of φ is denoted by $\langle \varphi'(u), v\rangle, \forall u, v \in X$. By $\to$ we denote the strong and by $\rightharpoonup$ the weak convergence. For a function u, u^+ denotes the functions $\max\{u(x), 0\}$. The symbol δ_{ij} denotes the Kronecker symbol: $\delta_{ij} = \begin{cases} 1, \ i = j \\ 0, \ i \neq j. \end{cases}$ We use $o(h)$ to mean $o(h)/|h| \to 0$ as $|h| \to 0$.

2 An equivalent equation for Eq. (1.1)

For $x \in \mathbb{R}^N$, let $y = |x|^{-b/2}x$. To u, a C^2 function in $\mathbb{R}^N$, we associate a function v, a C^2 function in $\mathbb{R}^N \setminus \{0\}$ by the transformation

$$u(x) = |x|^{-\frac{b}{4}(N-2)} v(|x|^{-\frac{b}{2}} x_1, \cdots, |x|^{-\frac{b}{2}} x_N). \tag{2.1}$$

Lemma 2.1. *Under the above assumptions,*

$$\triangle_x u(x) = |y|^{-\frac{b(N+2)}{2(2-b)}} \Big(\sum_{i,j=1}^N \frac{\partial}{\partial y_j}\Big(A_{ij}(y)\frac{\partial v}{\partial y_i}\Big) - \frac{C_b}{|y|^2} v\Big). \tag{2.2}$$

where

$$A_{ij}(y) = \delta_{ij} + (\frac{b^2}{4} - b)\frac{y_i y_j}{|y|^2}, \ i, j = 1, \cdots, N. \tag{2.3}$$

Proof. Let $r = |x|$. By direct computations,

$$\begin{aligned} \frac{\partial u}{\partial x_i} \quad = \quad & r^{-\frac{b(N-2)}{4} - \frac{b}{2}} \frac{\partial v}{\partial y_i} - \frac{b}{2} r^{-\frac{b(N-2)}{4} - \frac{b}{2} - 2} x_i \sum_{j=1}^N x_j \frac{\partial v}{\partial y_j} \\ & -\frac{b}{4}(N-2) r^{-\frac{b(N-2)}{4} - 2} x_i v \end{aligned} \tag{2.4}$$

and

$$\begin{aligned}\frac{\partial^2 u}{\partial x_i^2} &= -\frac{bN}{2}r^{-\frac{b(N-2)}{4}-\frac{b}{2}-2}x_i\frac{\partial v}{\partial y_i}+r^{-\frac{b(N-2)}{4}-b}\frac{\partial^2 v}{\partial y_i^2}-br^{-\frac{b(N-2)}{4}-b-2}\sum_{j=1}^N x_jx_i\frac{\partial^2 v}{\partial y_j\partial y_i}\\ &\quad+\Big(\frac{b^2}{4}(N-1)+b\Big)r^{-\frac{b(N-2)}{4}-\frac{b}{2}-4}x_i^2\sum_{j=1}^N x_j\frac{\partial v}{\partial y_j}\\ &\quad-\frac{b}{2}r^{-\frac{b(N-2)}{4}-\frac{b}{2}-2}\sum_{j=1}^N x_j\frac{\partial v}{\partial y_j}+\frac{b^2}{4}r^{-\frac{b(N-2)}{4}-b-4}x_i^2\sum_{j,k=1}^N x_jx_k\frac{\partial^2 v}{\partial y_j\partial y_k}\\ &\quad+\frac{b}{4}(N-2)(\frac{b}{4}(N-2)+2)r^{-\frac{b}{4}(N-2)-4}x_i^2v-\frac{b}{4}(N-2)r^{-\frac{b}{4}(N-2)-2}v.\end{aligned}$$

Then

$$\begin{aligned}\triangle_x u &= \sum_{i=1}^N\frac{\partial^2 u}{\partial x_i^2}\\ &= r^{-\frac{b(N-2)}{4}-b}\Big\{\triangle_y v+(\frac{b^2}{4}-b)r^{-2}\sum_{i,j=1}^N x_ix_j\frac{\partial^2 v}{\partial y_i\partial y_j}\\ &\quad+(\frac{b^2}{4}-b)(N-1)r^{\frac{b}{2}-2}\sum_{i=1}^N x_i\frac{\partial v}{\partial y_i}-\frac{b}{4}(1-\frac{b}{4})(N-2)^2r^{b-2}v\Big\}.\end{aligned}\tag{2.5}$$

Since $y=|x|^{-b/2}x$, we have $r=|y|^{\frac{2}{2-b}}$ and $x_i=|y|^{\frac{b}{2-b}}y_i$, $1\le i\le N$. Then

$$\begin{aligned}&r^{-2}\sum_{i,j=1}^N x_ix_j\frac{\partial^2 v}{\partial y_i\partial y_j}+(N-1)r^{\frac{b}{2}-2}\sum_{i=1}^N x_i\frac{\partial v}{\partial y_i}\\ =\;&|y|^{-2}\sum_{i,j=1}^N y_iy_j\frac{\partial^2 v}{\partial y_i\partial y_j}+(N-1)|y|^{-2}\sum_{i=1}^N y_i\frac{\partial v}{\partial y_i}\\ =\;&\sum_{i,j=1}^N\frac{\partial}{\partial y_j}\Big(\frac{y_iy_j}{|y|^2}\frac{\partial v}{\partial y_i}\Big).\end{aligned}\tag{2.6}$$

Substituting (2.6) and $r=|y|^{\frac{2}{2-b}}$ into (2.5) results in

$$\begin{aligned}\triangle_x u(x) &= |y|^{-\frac{b(N+2)}{2(2-b)}}\Big(\triangle_y v+(\frac{b^2}{4}-b)\sum_{i,j=1}^N\frac{\partial}{\partial y_j}\Big(\frac{y_iy_j}{|y|^2}\frac{\partial v}{\partial y_i}\Big)-\frac{C_b}{|y|^2}v\Big)\\ &= |y|^{-\frac{b(N+2)}{2(2-b)}}\Big(\sum_{i,j=1}^N\frac{\partial}{\partial y_j}\Big(A_{ij}(y)\frac{\partial v}{\partial y_i}\Big)-\frac{C_b}{|y|^2}v\Big).\end{aligned}$$

□

Let

$$H^1_{loc}(\mathbb{R}^N) = \{u \mid \text{for every bounded domain } \Omega\subset\mathbb{R}^N,\ \int_\Omega|\nabla u|^2dx+\int_\Omega u^2dx<+\infty\}.\tag{2.7}$$

From the classical Hardy inequality (see, e.g., [7, Lemma 2.1]), we deduce that for every bounded C^1 domain $\Omega\subset\mathbb{R}^N$, there exists $C_\Omega>0$ such that, for every $u\in H^1_{loc}(\mathbb{R}^N)$,

$$\int_\Omega\frac{u^2}{|x|^2}dx\le C_\Omega(\int_\Omega|\nabla u|^2dx+\int_\Omega u^2dx)\tag{2.8}$$

Theorem 2.2. *If $v \in H^1_{loc}(\mathbb{R}^N)$ is a weak solution of the equation*

$$-\sum_{i,j=1}^{N} \frac{\partial}{\partial y_j}\Big(A_{ij}(y)\frac{\partial v}{\partial y_i}\Big) + V_* v = K_*|v|^{p-2}v \text{ in } \mathbb{R}^N, \tag{2.9}$$

i.e., for every $\psi \in C_0^\infty(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N}\sum_{i,j=1}^{N} A_{ij}(y)\frac{\partial v}{\partial y_i}\frac{\partial \psi}{\partial y_j}dy + \int_{\mathbb{R}^N} V_*(y)v\psi dy = \int_{\mathbb{R}^N} K_*(y)|v|^{p-2}v\psi dy, \tag{2.10}$$

and u is defined by (2.1), then $u \in H^1_{loc}(\mathbb{R}^N)$ and it is a weak solution of (1.1), i.e., for every $\varphi \in C_0^\infty(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} \nabla u \nabla \varphi dx + \int_{\mathbb{R}^N} V(x)u\varphi dx = \int_{\mathbb{R}^N} K(x)|u|^{p-2}u\varphi dx. \tag{2.11}$$

Proof. Using the spherical coordinates

$$\begin{aligned}
&x_1 = r\cos\sigma_1,\\
&x_2 = r\sin\sigma_1\cos\sigma_2,\\
&\cdots\cdots\\
&x_j = r\sin\sigma_1\sin\sigma_2\cdots\sin\sigma_{j-1}\cos\sigma_j,\ 2\le j\le N-1,\\
&\cdots\cdots\\
&x_N = r\sin\sigma_1\sin\sigma_2\cdots\sin\sigma_{N-2}\sin\sigma_{N-1},
\end{aligned}$$

where $0 \le \sigma_j < \pi$, $j = 1, 2, \ldots, N-2$, $0 \le \sigma_{N-1} < 2\pi$, we have

$$dx = r^{N-1}f(\sigma)drd\sigma_1\cdots d\sigma_{N-1},$$

where $f(\sigma) = \sin^{N-2}\sigma_1 \sin^{N-3}\sigma_2\cdots\sin\sigma_{N-2}$. Recall that $y = |x|^{-\frac{b}{2}}x$. Let $R = |y|$. Then $r = R^{\frac{2}{2-b}}$ and

$$\begin{aligned}
dx &= r^{N-1}f(\sigma)drd\sigma_1\cdots d\sigma_{N-1} = R^{\frac{2(N-1)}{2-b}}f(\sigma)d(R^{\frac{2}{2-b}})d\sigma_1\cdots d\sigma_{N-1}\\
&= \frac{2}{2-b}R^{\frac{2N}{2-b}-1}f(\sigma)dRd\sigma_1\cdots d\sigma_{N-1} = \frac{2}{2-b}|y|^{\frac{bN}{2-b}}dy.
\end{aligned} \tag{2.12}$$

Here, we used $dy = R^{N-1}f(\sigma)dRd\sigma_1\cdots d\sigma_{N-1}$ in the above last inequality. From (2.4), (2.12) and (2.8), we deduce that there exists $C > 0$ such that for every bounded domain $\Omega \subset \mathbb{R}^N$,

$$\begin{aligned}
\int_\Omega |\frac{\partial u}{\partial x_i}|^2 dx \ &\le\ C\int_\Omega r^{-\frac{b(N-2)}{2}-b}\Big(\frac{\partial v}{\partial y_i}(|x|^{-b/2}x)\Big)^2 dx\\
&\quad + C\int_\Omega r^{-\frac{b(N-2)}{2}-b-4}\Big(x_i\sum_{j=1}^{N}x_j\frac{\partial v}{\partial y_j}(|x|^{-b/2}x)\Big)^2 dx\\
&\quad + C\int_\Omega r^{-\frac{b(N-2)}{2}-4}x_i^2 v^2(|x|^{-b/2}x)dx\\
&= \frac{2C}{2-b}\int_\Omega\Big(\frac{\partial v(y)}{\partial y_i}\Big)^2 dy + \frac{2C}{2-b}\int_\Omega\Big(\frac{y_i}{|y|}\sum_{j=1}^{N}\frac{y_j}{|y|}\frac{\partial v(y)}{\partial y_j}\Big)^2 dy\\
&\quad + \frac{2C}{2-b}\int_\Omega |y|^{-4}y_i^2 v^2(y)dy\\
&\le\ C''(\int_\Omega |\nabla v|^2 dy + \int_\Omega \frac{v^2}{|y|^2}dy) < +\infty.
\end{aligned}$$

Moreover,

$$\int_\Omega u^2dx=\int_\Omega |x|^{-\frac{b}{2}(N-2)}v^2(|x|^{-\frac{b}{2}}x)dx=\frac{2}{2-b}\int_\Omega |y|^{\frac{2b}{2-b}}v^2(y)dy<+\infty.$$

Therefore, $u\in H^1_{loc}(\mathbb{R}^N)$. Then, to prove u satisfies (2.11) for every $\varphi\in C_0^\infty(\mathbb{R}^N)$, it suffices to prove that (2.11) holds for every $\varphi\in C_0^\infty(\mathbb{R}^N\setminus\{0\})$. For $\varphi\in C_0^\infty(\mathbb{R}^N\setminus\{0\})$, let $\psi\in C_0^\infty(\mathbb{R}^N\setminus\{0\})$ be such that

$$\varphi(x)=|x|^{-\frac{b}{4}(N-2)}\psi(|x|^{-\frac{b}{2}}x).$$

By using the divergence theorem and Lemma 2.1, we get that

$$\begin{aligned}
&\int_{\mathbb{R}^N}\nabla u\nabla\varphi dx\\
=&-\int_{\mathbb{R}^N}u\triangle\varphi dx\\
=&-\int_{\mathbb{R}^N}u\cdot|y|^{-\frac{b(N+2)}{2(2-b)}}\Big(\sum_{i,j=1}^N\frac{\partial}{\partial y_j}\Big(A_{ij}(y)\frac{\partial\psi}{\partial y_i}\Big)-\frac{C_b}{|y|^2}\psi\Big)dx\\
=&-\int_{\mathbb{R}^N}|x|^{-\frac{b}{4}(N-2)}v(|x|^{-\frac{b}{2}}x)\cdot|y|^{-\frac{b(N+2)}{2(2-b)}}\Big(\sum_{i,j=1}^N\frac{\partial}{\partial y_j}\Big(A_{ij}(y)\frac{\partial\psi}{\partial y_i}\Big)-\frac{C_b}{|y|^2}\psi\Big)dx\\
=&-\int_{\mathbb{R}^N}|y|^{-\frac{b(N-2)}{2(2-b)}}v(y)\cdot|y|^{-\frac{b(N+2)}{2(2-b)}}\Big(\sum_{i,j=1}^N\frac{\partial}{\partial y_j}\Big(A_{ij}(y)\frac{\partial\psi}{\partial y_i}\Big)-\frac{C_b}{|y|^2}\psi\Big)\frac{2}{2-b}|y|^{\frac{bN}{2-b}}dy\\
=&-\frac{2}{2-b}\int_{\mathbb{R}^N}v\cdot\Big(\sum_{i,j=1}^N\frac{\partial}{\partial y_j}\Big(A_{ij}(y)\frac{\partial\psi}{\partial y_i}\Big)-\frac{C_b}{|y|^2}\psi\Big)dy\\
=&\frac{2}{2-b}\int_{\mathbb{R}^N}\sum_{i,j=1}^N A_{ij}(y)\frac{\partial v}{\partial y_i}\frac{\partial\psi}{\partial y_j}dy-\frac{2C_b}{2-b}\int_{\mathbb{R}^N}\frac{v\psi}{|y|^2}dy.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\int_{\mathbb{R}^N}V(x)u\varphi dx\\
=&\frac{2}{2-b}\int_{\mathbb{R}^N}V(|y|^{\frac{b}{2-b}}y)u(|y|^{\frac{b}{2-b}}y)\varphi(|y|^{\frac{b}{2-b}}y)|y|^{\frac{bN}{2-b}}dy\\
=&\frac{2}{2-b}\int_{\mathbb{R}^N}|y|^{\frac{2b}{2-b}}V(|y|^{\frac{b}{2-b}}y)\cdot|y|^{\frac{b(N-2)}{2(2-b)}}u(|y|^{\frac{b}{2-b}}y)\cdot|y|^{\frac{b(N-2)}{2(2-b)}}\varphi(|y|^{\frac{b}{2-b}}y)dy\\
=&\frac{2}{2-b}\int_{\mathbb{R}^N}|y|^{\frac{2b}{2-b}}V(|y|^{\frac{b}{2-b}}y)v(y)\psi(y)dy
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^N}K(x)|u|^{p-2}u\varphi dx\\
=&\int_{\mathbb{R}^N}K(|y|^{\frac{b}{2-b}}y)|u(|y|^{\frac{b}{2-b}}y)|^{p-2}u(|y|^{\frac{b}{2-b}}y)\varphi(|y|^{\frac{b}{2-b}}y)\frac{2}{2-b}|y|^{\frac{bN}{2-b}}dy\\
=&\frac{2}{2-b}\int_{\mathbb{R}^N}|y|^{\frac{2s}{2-b}}K(|y|^{\frac{b}{2-b}}y)|v(y)|^{p-2}v(y)\psi(y)dy.
\end{aligned}$$

Therefore,

$$\int_{\mathbb{R}^N}\nabla u\nabla\varphi dx+\int_{\mathbb{R}^N}V(x)u\varphi dx-\int_{\mathbb{R}^N}K(x)|u|^{p-2}u\varphi dx$$

$$
\begin{aligned}
&= \frac{2}{2-b}\Big(\int_{\mathbb{R}^N}\sum_{i,j=1}^N A_{ij}(y)\frac{\partial v}{\partial y_i}\frac{\partial \psi}{\partial y_j}dy - C_b\int_{\mathbb{R}^N}\frac{v\psi}{|y|^2}dy \\
&\quad + \int_{\mathbb{R}^N}|y|^{\frac{2b}{2-b}}V(|y|^{\frac{b}{2-b}}y)v(y)\psi(y)dy \\
&\quad - \int_{\mathbb{R}^N}|y|^{\frac{2s}{2-b}}K(|y|^{\frac{b}{2-b}}y)|v(y)|^{p-2}v(y)\psi(y)dy\Big) \\
&= \frac{2}{2-b}\Big(\int_{\mathbb{R}^N}\sum_{i,j=1}^N A_{ij}(y)\frac{\partial v}{\partial y_i}\frac{\partial \psi}{\partial y_j}dy + \int_{\mathbb{R}^N}V_*(y)v\psi dy - \int_{\mathbb{R}^N}K_*(y)|v|^{p-2}v\psi dy\Big) \\
&= 0.
\end{aligned}
$$

This completes the proof. □

This theorem implies that the problem of looking for solutions of (1.1) can be reduced to a problem of looking for solutions of (2.9).

3 The variational functional for Eq. (2.9).

The following inequality is a variant Hardy inequality.

Lemma 3.1. *If $v \in H^1(\mathbb{R}^N)$, then*

$$
\int_{\mathbb{R}^N}\frac{|x\cdot\nabla v|^2}{|x|^2}dx \geq \frac{(N-2)^2}{4}\int_{\mathbb{R}^N}\frac{|v|^2}{|x|^2}dx. \tag{3.1}
$$

Proof. We only give the proof of (3.1) for $v \in C_0^\infty(\mathbb{R}^N)$, since $C_0^\infty(\mathbb{R}^N)$ is dense in $H^1(\mathbb{R}^N)$. For $v \in C_0^\infty(\mathbb{R}^N)$, we have the following identity

$$
|v(x)|^2 = -\int_1^\infty \frac{d}{d\lambda}|v(\lambda x)|^2 d\lambda = -2\int_1^\infty v(\lambda x)\cdot(x\cdot\nabla v(\lambda x))d\lambda.
$$

By using the Hölder inequality, it follows that

$$
\begin{aligned}
\int_{\mathbb{R}^N}\frac{|v(x)|^2}{|x|^2}dx &= -2\int_1^\infty\int_{\mathbb{R}^N}\frac{v(\lambda x)}{|x|^2}\cdot(x\cdot\nabla v(\lambda x))dxd\lambda \\
&= -2\int_1^\infty\frac{d\lambda}{\lambda^{N-1}}\int_{\mathbb{R}^N}\frac{v(x)}{|x|^2}\cdot(x\cdot\nabla v(x))dx \\
&= -\frac{2}{N-2}\int_{\mathbb{R}^N}\frac{v(x)}{|x|^2}\cdot(x\cdot\nabla v(x))dx \\
&\leq \frac{2}{N-2}\Big(\int_{\mathbb{R}^N}\frac{v^2(x)}{|x|^2}dx\Big)^{1/2}\Big(\int_{\mathbb{R}^N}\frac{|x\cdot\nabla v|^2}{|x|^2}dx\Big)^{1/2}.
\end{aligned}
$$

And then we conclude that

$$
\int_{\mathbb{R}^N}\frac{|x\cdot\nabla v|^2}{|x|^2}dx \geq \frac{(N-2)^2}{4}\int_{\mathbb{R}^N}\frac{|v|^2}{|x|^2}dx.
$$

□

From the definition of $A_{ij}(x)$ (see (2.3)), it is easy to verify that, for $u \in H^1(\mathbb{R}^N)$,

$$
\int_{\mathbb{R}^N}\sum_{i,j=1}^N A_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}dx = \int_{\mathbb{R}^N}|\nabla u|^2dx + \Big(\frac{b^2}{4}-b\Big)\int_{\mathbb{R}^N}\frac{|x\cdot\nabla u|^2}{|x|^2}dx. \tag{3.2}
$$

Lemma 3.2. *There exist constants $C_1 > 0$ and $C_2 > 0$ such that, for every $u \in H^1(\mathbb{R}^N)$,*

$$C_1||u||^2 \leq \int_{\mathbb{R}^N} |\nabla u|^2 dx + (\frac{b^2}{4} - b) \int_{\mathbb{R}^N} \frac{|x \cdot \nabla u|^2}{|x|^2} dx + \int_{\mathbb{R}^N} V_*(x)|u|^2 dx \leq C_2||u||^2.$$

Proof. From the conditions $(\mathbf{A_1})$ and $(\mathbf{A_2})$, we deduce that there exists a constant $C > 0$ such that

$$|x|^{\frac{2b}{2-b}} V(|x|^{\frac{b}{2-b}} x) \leq C(1 + |x|^{-2}), \ \forall x \in \mathbb{R}^N \setminus \{0\}. \tag{3.3}$$

Since

$$\int_{\mathbb{R}^N} V_*(x)|u|^2 dx = \int_{\mathbb{R}^N} |x|^{\frac{2b}{2-b}} V(|x|^{\frac{b}{2-b}} x)|u|^2 dx + C_b \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} dx,$$

by (3.3) and the classical Hardy inequality (see, e.g., [7])

$$\frac{(N-2)^2}{4} \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} dx \leq \int_{\mathbb{R}^N} |\nabla u|^2 dx, \ \forall u \in H^1(\mathbb{R}^N),$$

we deduce that there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}^N} V_*(x)|u|^2 dx \leq C||u||^2.$$

This together with the fact that $\int_{\mathbb{R}^N} \frac{|x \cdot \nabla u|^2}{|x|^2} dx \leq \int_{\mathbb{R}^N} |\nabla u|^2 dx$ yields that there exists a constant $C_2 > 0$ such that

$$\begin{aligned} & \int_{\mathbb{R}^N} |\nabla u|^2 dx + (\frac{b^2}{4} - b) \int_{\mathbb{R}^N} \frac{|x \cdot \nabla u|^2}{|x|^2} dx + \int_{\mathbb{R}^N} V_*(x)|u|^2 dx \\ \leq \ & C_2||u||^2, \ \forall u \in H^1(\mathbb{R}^N). \end{aligned} \tag{3.4}$$

If $0 < b < 2$, then $\frac{b^2}{4} - b < 0$ and

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla u|^2 dx + (\frac{b^2}{4} - b) \int_{\mathbb{R}^N} \frac{|x \cdot \nabla u|^2}{|x|^2} dx &\geq \int_{\mathbb{R}^N} |\nabla u|^2 dx + (\frac{b^2}{4} - b) \int_{\mathbb{R}^N} |\nabla u|^2 dx \\ &= (1 - b/2)^2 \int_{\mathbb{R}^N} |\nabla u|^2 dx. \end{aligned} \tag{3.5}$$

In this case, $C_b = \frac{b}{4}(1 - \frac{b}{4})(N-2)^2 > 0$ and

$$\begin{aligned} \int_{\mathbb{R}^N} V_*(x)|u|^2 dx &= \int_{\mathbb{R}^N} |x|^{\frac{2b}{2-b}} V(|x|^{\frac{b}{2-b}} x)|u|^2 dx + C_b \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} dx \\ &\geq \int_{\mathbb{R}^N} |x|^{\frac{2b}{2-b}} V(|x|^{\frac{b}{2-b}} x)|u|^2 dx. \end{aligned} \tag{3.6}$$

The conditions $(\mathbf{A_1})$ and $(\mathbf{A_2})$ imply that there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} |x|^{\frac{2b}{2-b}} V(|x|^{\frac{b}{2-b}} x) u^2 dx \geq C \int_{\mathbb{R}^N} u^2 dx. \tag{3.7}$$

Combining $(3.5) - (3.7)$ yields that there exists a constant $C_1 > 0$ such that

$$\begin{aligned} & \int_{\mathbb{R}^N} |\nabla u|^2 dx + (\frac{b^2}{4} - b) \int_{\mathbb{R}^N} \frac{|x \cdot \nabla u|^2}{|x|^2} dx + \int_{\mathbb{R}^N} V_*(x)|u|^2 dx \\ \geq \ & C_1||u||^2, \ \forall u \in H^1(\mathbb{R}^N). \end{aligned} \tag{3.8}$$

If $b < 0$, (3.7) still holds. From Lemma 3.1 and (3.7), we deduce that there exists a constant $C_1 > 0$ such that, for every $u \in H^1(\mathbb{R}^N)$,

$$\begin{aligned} & \int_{\mathbb{R}^N} |\nabla u|^2 dx + (\frac{b^2}{4} - b) \int_{\mathbb{R}^N} \frac{|x \cdot \nabla u|^2}{|x|^2} dx + \int_{\mathbb{R}^N} V_*(x)|u|^2 dx \\ = & \int_{\mathbb{R}^N} |\nabla u|^2 dx + (\frac{b^2}{4} - b)\Big(\int_{\mathbb{R}^N} \frac{|x \cdot \nabla u|^2}{|x|^2} dx - \frac{(N-2)^2}{4} \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} dx \Big) \\ & + \int_{\mathbb{R}^N} |x|^{\frac{2b}{2-b}} V(|x|^{\frac{b}{2-b}} x)|u|^2 dx \\ \geq & \int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} |x|^{\frac{2b}{2-b}} V(|x|^{\frac{b}{2-b}} x)|u|^2 dx \geq C_1 ||u||^2. \end{aligned} \tag{3.9}$$

Then the desired result of this lemma follows from (3.4), (3.8) and (3.9) immediately. □

This lemma implies that

$$||u||_A = (\int_{\mathbb{R}^N} |\nabla u|^2 dx + (\frac{b^2}{4} - b) \int_{\mathbb{R}^N} \frac{|x \cdot \nabla u|^2}{|x|^2} dx + \int_{\mathbb{R}^N} V_*(x)|u|^2 dx)^{1/2} \tag{3.10}$$

is equivalent to the standard norm $||\cdot||$ in $H^1(\mathbb{R}^N)$. We denote the inner product associated with $||\cdot||_A$ by $(\cdot,\cdot)_A$, i.e.,

$$\begin{aligned} (u,v)_A \quad = & \int_{\mathbb{R}^N} \nabla u \nabla v dx + \int_{\mathbb{R}^N} V_*(x) uv dx \\ & + (\frac{b^2}{4} - b) \int_{\mathbb{R}^N} \frac{(x \cdot \nabla u)(x \cdot \nabla v)}{|x|^2} dx. \end{aligned} \tag{3.11}$$

By the Sobolev inequality, we have

$$S_A := \inf_{u \in H^1(\mathbb{R}^N) \setminus \{0\}} \frac{||u||_A^2}{(\int_{\mathbb{R}^N} |u|^p dx)^{2/p}} > 0 \tag{3.12}$$

and

$$||u||_A \geq S_A^{\frac{1}{2}} (\int_{\mathbb{R}^N} |u|^p dx)^{1/p}, \ \forall u \in H^1(\mathbb{R}^N). \tag{3.13}$$

By the condition $(\mathbf{A_1})$ and $(\mathbf{A_2})$, if $0 < b < 2$, then K_* is bounded in $\mathbb{R}^N$. Therefore, by (3.13), there exists $C > 0$ such that

$$(\int_{\mathbb{R}^N} K_*(x)(u^+)^p dx)^{1/p} \leq C||u||_A, \ \forall u \in H^1(\mathbb{R}^N). \tag{3.14}$$

However, if $b < 0$, K_* has a singularity at $x = 0$, i.e.,

$$K_*(x) \sim |x|^{\frac{2s}{2-b}} K(0), \ \text{as } |x| \to 0. \tag{3.15}$$

Recall that $p = 2(N - 2s/b)/(N-2)$ and $2s/(2-b) > -2s/b$ if $b < 0$. Then by the Hardy-Sobolev inequality (see, for example, [8, Lemma 3.2]), we deduce that there exists $C > 0$ such that (3.14) still holds. Therefore, the functional

$$J(u) = \frac{1}{2}||u||_A^2 - \frac{1}{p} \int_{\mathbb{R}^N} K_*(x)(u^+)^p dx, \ u \in H^1(\mathbb{R}^N) \tag{3.16}$$

is a C^2 functional defined in $H^1(\mathbb{R}^N)$. Moreover, it easy to check that the Gateaux derivative of J is

$$\langle J'(u), h \rangle = (u,h)_A - \int_{\mathbb{R}^N} K_*(x)(u^+)^{p-1} h dx, \ \forall u, h \in H^1(\mathbb{R}^N)$$

and the critical points of J are nonnegative solutions of (2.9).

4 Some minimizing problems

For $\theta = (\theta_1, \cdots, \theta_N) \in \mathbb{R}^N$ with $|\theta| = 1$, let

$$B_{ij}(\theta) = \delta_{ij} + (\frac{b^2}{4} - b)\theta_i\theta_j,\ i, j = 1, \cdots, N. \tag{4.1}$$

By this definition, we have, for $u \in H^1(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} \sum_{i,j=1}^{N} B_{ij}(\theta)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} dx = \int_{\mathbb{R}^N} |\nabla u|^2 dx + (\frac{b^2}{4} - b)\int_{\mathbb{R}^N} |\theta \cdot \nabla u|^2 dx. \tag{4.2}$$

From

$$\begin{aligned} (1 + |\frac{b^2}{4} - b|)\int_{\mathbb{R}^N} |\nabla u|^2 dx &\geq \int_{\mathbb{R}^N} |\nabla u|^2 dx + (\frac{b^2}{4} - b)\int_{\mathbb{R}^N} |\theta \cdot \nabla u|^2 dx \\ &\geq \left\{ \begin{array}{l} (1 - b/2)^2 \int_{\mathbb{R}^N} |\nabla u|^2 dx,\ 0 < b < 2 \\ \int_{\mathbb{R}^N} |\nabla u|^2 dx, \quad b < 0, \end{array} \right. \end{aligned}$$

we deduce that the norm defined by

$$||u||_\theta := (\int_{\mathbb{R}^N} |\nabla u|^2 dx + (\frac{b^2}{4} - b)\int_{\mathbb{R}^N} |\theta \cdot \nabla u|^2 dx + a\int_{\mathbb{R}^N} |u|^2 dx)^{1/2} \tag{4.3}$$

is equivalent to the standard norm $||\cdot||$ in $H^1(\mathbb{R}^N)$. The inner product corresponding to $||\cdot||_\theta$ is

$$(u, v)_\theta = \int_{\mathbb{R}^N} \nabla u \nabla v dx + a\int_{\mathbb{R}^N} uv dx + (\frac{b^2}{4} - b)\int_{\mathbb{R}^N} (\theta \cdot \nabla u)(\theta \cdot \nabla v) dx.$$

Lemma 4.1. *The infimum*

$$\inf_{u \in H^1(\mathbb{R}^N)\setminus\{0\}} \frac{||u||_\theta^2}{(\int_{\mathbb{R}^N} |u|^p dx)^{2/p}} \tag{4.4}$$

is independent of $\theta \in \mathbb{R}^N$ *with* $|\theta| = 1$.

Proof. In this proof, we always view a vector in $\mathbb{R}^N$ as a $1 \times N$ matrix. And we use A^T to denote the conjugate matrix of a matrix A.

For any $\theta, \theta' \in \mathbb{R}^N$ with $|\theta| = |\theta'| = 1$, let G be an $N \times N$ orthogonal matrix such that $\theta' \cdot G^T = \theta$. For any $u \in H^1(\mathbb{R}^N)$, let $v(x) = u(xG)$, $x \in \mathbb{R}^N$. The assumption G is an $N \times N$ orthogonal matrix implies that $GG^T = I$, where I is the $N \times N$ identity matrix. Then it is easy to check that

$$\int_{\mathbb{R}^N} |v|^2 dx = \int_{\mathbb{R}^N} |u|^2 dx,\ \int_{\mathbb{R}^N} |v|^p dx = \int_{\mathbb{R}^N} |u|^p dx. \tag{4.5}$$

Note that

$$\nabla v(x) = (\nabla u)(xG) \cdot G. \tag{4.6}$$

By $GG^T = I$, we have

$$\begin{aligned} |\nabla v(x)|^2 &= \nabla v(x) \cdot (\nabla v(x))^T \\ &= (\nabla u)(xG) \cdot G \cdot G^T \cdot ((\nabla u)(xG))^T = |(\nabla u)(xG)|^2. \end{aligned}$$

It follows that

$$\int_{\mathbb{R}^N} |\nabla v(x)|^2 dx = \int_{\mathbb{R}^N} |(\nabla u)(xG)|^2 dx = \int_{\mathbb{R}^N} |\nabla u(x)|^2 dx. \tag{4.7}$$

By (4.6) and $\theta' \cdot G^T = \theta$, we get that

$$\begin{aligned}
& \sum_{i=1}^{N} \theta_i' \frac{\partial v}{\partial x_i} = \theta' \cdot ((\nabla u)(xG) \cdot G)^T = \theta' \cdot G^T \cdot ((\nabla u)(xG))^T = \theta \cdot ((\nabla u)(xG))^T \\
= \ & \sum_{i=1}^{N} \theta_i (\frac{\partial u}{\partial y_i})(xG).
\end{aligned}$$

It follows that

$$\begin{aligned}
\int_{\mathbb{R}^N} |\theta' \cdot \nabla v|^2 dx &= \int_{\mathbb{R}^N} |\sum_{i=1}^{N} \theta_i' \frac{\partial v}{\partial x_i}|^2 dx \\
&= \int_{\mathbb{R}^N} |\sum_{i=1}^{N} \theta_i (\frac{\partial u}{\partial y_i})(xG)|^2 dx \\
&= \int_{\mathbb{R}^N} |\sum_{i=1}^{N} \theta_i \frac{\partial u}{\partial x_i}|^2 dx = \int_{\mathbb{R}^N} |\theta \cdot \nabla u|^2 dx. \qquad (4.8)
\end{aligned}$$

By (4.5), (4.7) and (4.8), we get that $||v||_{\theta'}^2 = ||u||_{\theta}^2$. This together with (4.5) leads to the result of this lemma. □

Since the infimum (4.4) is independent of $\theta \in \mathbb{R}^N$ with $|\theta| = 1$, we denote it by S.

Lemma 4.2. *Let S_p be the infimum in (1.9). Then $S = (1 - b/2)^{\frac{p-2}{p}} S_p$.*

Proof. Choosing $\theta = (1, 0, \cdots, 0)$ in $|| \cdot ||_\theta$, we have

$$||u||_\theta^2 = (1 - \frac{b}{2})^2 \int_{\mathbb{R}^N} |\frac{\partial u}{\partial x_1}|^2 dx + \sum_{i=2}^{N} \int_{\mathbb{R}^N} |\frac{\partial u}{\partial x_i}|^2 dx + a \int_{\mathbb{R}^N} u^2 dx.$$

By Lemma 4.1, we have

$$S = \inf_{u \in H^1(\mathbb{R}^N) \setminus \{0\}} \frac{(1 - \frac{b}{2})^2 \int_{\mathbb{R}^N} |\frac{\partial u}{\partial x_1}|^2 dx + \sum_{i=2}^{N} \int_{\mathbb{R}^N} |\frac{\partial u}{\partial x_i}|^2 dx + a \int_{\mathbb{R}^N} u^2 dx}{(\int_{\mathbb{R}^N} |u|^p dx)^{2/p}}.$$

Let

$$v(x) = u((1 - b/2)x_1, x_2, \cdots, x_N), \ x \in \mathbb{R}^N.$$

Then

$$\begin{aligned}
& \frac{(1 - \frac{b}{2})^2 \int_{\mathbb{R}^N} |\frac{\partial u}{\partial x_1}|^2 dx + \sum_{i=2}^{N} \int_{\mathbb{R}^N} |\frac{\partial u}{\partial x_i}|^2 dx + a \int_{\mathbb{R}^N} u^2 dx}{(\int_{\mathbb{R}^N} |u|^p dx)^{2/p}} \\
= \ & (1 - b/2)^{\frac{p-2}{p}} \frac{\int_{\mathbb{R}^N} |\nabla v|^2 dx + a \int_{\mathbb{R}^N} v^2 dx}{(\int_{\mathbb{R}^N} |v|^p dx)^{2/p}}.
\end{aligned}$$

It follows that

$$S = (1 - b/2)^{\frac{p-2}{p}} \inf_{v \in H^1(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla v|^2 dx + a \int_{\mathbb{R}^N} v^2 dx}{(\int_{\mathbb{R}^N} |v|^p dx)^{2/p}} = (1 - b/2)^{\frac{p-2}{p}} S_p.$$

□

Since the functionals $||u||_\theta^2$ and $\int_{\mathbb{R}^N} |u|^p dx$ are invariant by translations, the same argument as the proof of [14, Theorem 1.34] yields that there exists a positive minimizer U_θ for the infimum S. And from the Lagrange multiplier rule, it is a solution of

$$-\sum_{i,j=1}^{N} \frac{\partial}{\partial y_j} \Big(B_{ij}(\theta) \frac{\partial u}{\partial y_i} \Big) + au = S(u^+)^{p-1} \text{ in } \mathbb{R}^N$$

and $(\mu/S)^{-1/(p-2)}U_\theta$ is a solution of

$$-\sum_{i,j=1}^{N}\frac{\partial}{\partial y_j}\Big(B_{ij}(\theta)\frac{\partial u}{\partial y_i}\Big)+au=\mu(u^+)^{p-1} \text{ in } \mathbb{R}^N. \tag{4.9}$$

In the next section, we shall show that Eq.(4.9) is the "limit" equation of

$$-\sum_{i,j=1}^{N}\frac{\partial}{\partial y_j}\Big(A_{ij}(x)\frac{\partial u}{\partial y_i}\Big)+V_*(x)u=K_*(x)(u^+)^{p-1} \text{ in } \mathbb{R}^N. \tag{4.10}$$

It is easy to verify that

$$J_\theta(u)=\frac{1}{2}||u||_\theta^2-\frac{\mu}{p}\int_{\mathbb{R}^N}(u^+)^p dx,\ u\in H^1(\mathbb{R}^N), \tag{4.11}$$

is a C^2 functional defined in $H^1(\mathbb{R}^N)$, the Gateaux derivative of J_θ is

$$\langle J_\theta'(u),h\rangle=(u,h)_\theta-\mu\int_{\mathbb{R}^N}(u^+)^{p-1}h dx,\ \forall u,h\in H^1(\mathbb{R}^N).$$

and the critical points of this functional are solutions of (4.9).

Lemma 4.3. *Let $\theta\in\mathbb{R}^N$ satisfy $|\theta|=1$. If $u\neq 0$ is a critical point of J_θ, then*

$$J_\theta(u)\geq(\frac{1}{2}-\frac{1}{p})\mu^{-\frac{2}{p-2}}S^{\frac{p}{p-2}}. \tag{4.12}$$

Proof. Since u is a critical point of J_θ, we have

$$0=\langle J_\theta'(u),u\rangle=||u||_\theta^2-\mu\int_{\mathbb{R}^N}(u^+)^p dx. \tag{4.13}$$

It follows that

$$J_\theta(u)=(\frac{1}{2}-\frac{1}{p})\mu\int_{\mathbb{R}^N}(u^+)^p dx. \tag{4.14}$$

Since $u\neq 0$, by $||u||_\theta^2=\mu\int_{\mathbb{R}^N}(u^+)^p dx$ and $||u||_\theta^2\geq S(\int_{\mathbb{R}^N}(u^+)^p dx)^{2/p}$, we get that

$$\int_{\mathbb{R}^N}(u^+)^p dx\geq(S/\mu)^{p/(p-2)}.$$

This together with (4.14) yields the result of this lemma. □

5 Palais-Smale conditions for the functional J.

Recall that J is the functional defined by (3.16). By a $(PS)_c$ sequence of J, we mean a sequence $\{u_n\}\subset H^1(\mathbb{R}^N)$ such that $J(u_n)\to c$ and $J'(u_n)\to 0$ in $H^{-1}(\mathbb{R}^N)$ as $n\to\infty$, where $H^{-1}(\mathbb{R}^N)$ denotes the dual space of $H^1(\mathbb{R}^N)$. J is called satisfying $(PS)_c$ condition if every $(PS)_c$ sequence of J contains a convergent subsequence in $H^1(\mathbb{R}^N)$.

Our main result in this section reads as follows:

Theorem 5.1. *Under the assumptions $(\mathbf{A_1})$ and $(\mathbf{A_2})$, let $\{u_n\}\subset H^1(\mathbb{R}^N)$ be a $(PS)_c$ sequence of J. Then replacing $\{u_n\}$ if necessary by a subsequence, there exist a solution $u_0\in H^1(\mathbb{R}^N)$ of Eq.(4.10), a finite sequence $\{\theta_l\in\mathbb{R}^N\mid|\theta_l|=1,\ 1\leq l\leq k\}$, k functions $\{u_l\mid 1\leq i\leq k\}\subset H^1(\mathbb{R}^N)$ and k sequences $\{y_n^l\}\subset\mathbb{R}^N$ satisfying:*

(i). $-\sum_{i,j=1}^{N}\frac{\partial}{\partial y_j}\left(B_{ij}(\theta_l)\frac{\partial u_l}{\partial y_i}\right)+au_l=\mu(u_l^+)^{p-1}$ *in* $\mathbb{R}^N$,

(ii). $|y_n^l|\to\infty,\ |y_n^l-y_n^{l'}|\to\infty,\ l\neq l',\ n\to\infty$,

(iii). $||u_n-u_0-\sum_{l=1}^{k}u_l(\cdot-y_n^l)||\to 0$,

(iv). $J(u_0)+\sum_{i=1}^{l}J_{\theta_l}(u_l)=c$.

This theorem gives a precise representation of $(PS)_c$ sequence for the functional J. Through it, partial compactness for J can be regained (see Corollary 5.8).

To prove this theorem, we need some lemmas. Our proof of this theorem is inspired by the proof of [14, Theorem 8.4].

Lemma 5.2. *Let $u\in H^1(\mathbb{R}^N)$. Then for any sequence $\{y_n\}\subset\mathbb{R}^N$,*

$$\lim_{R\to\infty}\sup_n\int_{|x|>R}K_*(x+y_n)|u|^p dx=0.$$

If $|y_n|\to\infty$, $n\to\infty$, then

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|K_*(x+y_n)-\mu|\cdot|u|^p dx=0.$$

Proof. If $2>b>0$, then K_* is bounded in $\mathbb{R}^N$. In this case, the result of this lemma is obvious. If $b<0$, then $K_*(x)\sim|x|^{\frac{2s}{2-b}}K(0)$ as $|x|\to 0$. Since $2s/(2-b)>-2s/b$, by Lemma 3.2 of [8], the map $v\mapsto K_*^{1/p}v$ from $H^1(\mathbb{R}^N)\to L^p_{loc}(\mathbb{R}^N)$ is compact. Therefore, for any $\epsilon>0$, there exists $\delta_\epsilon>0$ such that

$$\sup_n\int_{|x|\leq\delta_\epsilon}K_*(x)|u(x-y_n)|^p dx\leq\epsilon.$$

And there exists $D(\epsilon)>0$ depending only on ϵ such that $K_*(x)\leq D(\epsilon),\ |x|\geq\delta_\epsilon$. Then for every n,

$$\begin{aligned}
&\int_{|x|>R}K_*(x+y_n)|u|^p dx\\
&\leq\int_{\{x\ |\ |x+y_n|\leq\delta_\epsilon,\ |x|>R\}}K_*(x+y_n)|u|^p dx+\int_{\{x\ |\ |x+y_n|>\delta_\epsilon,\ |x|>R\}}K_*(x+y_n)|u|^p dx\\
\leq\ &\epsilon+C(\epsilon)\int_{|x|>R}|u|^p dx.
\end{aligned}$$

It follows that $\limsup_{R\to\infty}\sup_n\int_{|x|>R}K_*(x+y_n)|u|^p dx\leq\epsilon$. Now let $\epsilon\to 0$.

Using the same argument in the above, for any $\epsilon>0$, there exist δ_ϵ and $D(\epsilon)$ such that

$$\sup_n\int_{|x+y_n|\leq\delta_\epsilon}|K_*(x+y_n)-\mu|\cdot|u|^p dx\leq\epsilon$$

and

$$|K_*(x+y_n)-\mu|\cdot|u|^p dx\leq(D(\epsilon)+\mu)|u|^p,\ |x+y_n|\geq\delta_\epsilon.$$

Since $y_n\to\infty$, we have $\lim K_*(x+y_n)=\mu$. Then using the Lebesgue theorem and the above two inequalities, we get that

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N}|K_*(x+y_n)-\mu|\cdot|u|^p dx\leq\epsilon.$$

Let $\epsilon\to 0$. Then we get the desired result of this lemma. □

Lemma 5.3. *Let $\rho > 0$. If $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$ and if*

$$\sup_{y\in\mathbb{R}^N}\int_{B(y,\rho)}|u_n|^2dx \to 0,\ n\to\infty, \tag{5.1}$$

then $k_*^{1/p}u_n \to 0$ in $L^p(\mathbb{R}^N)$.

Proof. Since $2s/(2-b) > -2s/b$, by Lemma 3.2 of [8], the map $v \mapsto K_*^{1/p}v$ from $H^1(\mathbb{R}^N)\to L^p_{loc}(\mathbb{R}^N)$ is compact. Therefore, for any $\epsilon > 0$, there exists $\delta_\epsilon > 0$ such that

$$\sup_n\int_{|x|\le\delta_\epsilon}K_*(x)|u_n|^pdx \le \epsilon.$$

And there exists $D(\epsilon) > 0$ depending only on ϵ such that $K_*(x)\le D(\epsilon),\ |x|\ge\delta_\epsilon$. By (5.1) and the Lions Lemma (see, for example, [14, Lemma 1.21]), we get that

$$\int_{|x|\ge\delta_\epsilon}K_*(x)|u_n|^pdx \le D(\epsilon)\int_{\mathbb{R}^N}|u_n|^pdx\to 0,\ n\to\infty.$$

Therefore,$\limsup_{n\to\infty}\int_{\mathbb{R}^N}K_*(x)|u_n|^pdx\le\epsilon$. Now let $\epsilon\to 0$. □

Lemma 5.4. *Let $\{y_n\}\subset\mathbb{R}^N$. If $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$, then*

$$K_*(x+y_n)(u_n^+)^{p-1} - K_*(x+y_n)((u_n-u)^+)^{p-1} - K_*(x+y_n)(u^+)^{p-1}\to 0 \text{ in } H^{-1}(\mathbb{R}^N).$$

One can follow the proof of [14, Lemma 8.1] step by step and use Lemma 5.2 to give the proof of this lemma.

The following Lemma is a variant Brézis-Lieb Lemma (see [4]) and its proof is similar to that of [14, Lemma 1.32].

Lemma 5.5. *Let $\{u_n\}\subset H^1(\mathbb{R}^N)$ and $\{y_n\}\subset\mathbb{R}^N$. If*

a) $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$,

b) $u_n\to u$ a.e. on $\mathbb{R}^N$, then

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}K_*(x+y_n)\cdot|(u_n^+)^p-((u_n-u)^+)^p-(u^+)^p|dx = 0.$$

Proof. Let $j(t) = \begin{cases} t^p,\ t\ge 0\\ 0,\ \ t<0\end{cases}$. Then j is a convex function. From [4, Lemma 3], we have for any $\epsilon>0$, there exists $C(\epsilon)>0$ such that for all $a,b\in\mathbb{R}$,

$$|j(a+b)-j(b)|\le\epsilon j(a)+C(\epsilon)j(b). \tag{5.2}$$

Hence

$$\begin{aligned} f_n^\epsilon &:= \Big(K_*(x+y_n)\cdot|(u_n^+)^p-((u_n-u)^+)^p-(u^+)^p| - \epsilon K_*(x+y_n)\cdot((u_n-u)^+)^p\Big)^+\\ &\le (1+C(\epsilon))K_*(x+y_n)\cdot(u^+)^p. \end{aligned}$$

By Lemma 3.2 of [8], the map $v\mapsto K_*^{1/p}v$ from $H^1(\mathbb{R}^N)\to L^p_{loc}(\mathbb{R}^N)$ is compact. We get that there exists $\delta_\epsilon>0$ such that for any n,

$$\int_{|x+y_n|<\delta_\epsilon}f_n^\epsilon dx<\epsilon. \tag{5.3}$$

And there exists $D(\epsilon) > 0$ depending only on ϵ such that $K_*(x) \leq D(\epsilon),\ |x| \geq \delta_\epsilon$. Then

$$f_n^\epsilon \leq (1 + C(\epsilon))D(\epsilon) \cdot (u^+)^p,\ |x + y_n| \geq \delta_\epsilon.$$

By the Lebesgue theorem, $\int_{|x+y_n|\geq\delta_\epsilon} f_n^\epsilon dx \to 0,\ n \to \infty$. This together with (5.3) yields

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} f_n^\epsilon dx \leq \epsilon.$$

The left proof is the same as the proof of [14, Lemma 1.32]. □

Lemma 5.6. *If*

$$\begin{aligned}
&u_n \rightharpoonup u \text{ in } H^1(\mathbb{R}^N),\\
&u_n \to u \text{ a.e. on } \mathbb{R}^N,\\
&J(u_n) \to c,\\
&J'(u_n) \to 0 \text{ in } H^{-1}(\mathbb{R}^N),
\end{aligned}$$

then $J'(u) = 0$ *in* $H^{-1}(\mathbb{R}^N)$ *and* $v_n := u_n - u$ *is such that*

$$\begin{aligned}
&||v_n||_A^2 = ||u_n||_A^2 - ||u||_A^2 + o(1),\\
&J(v_n) \to c - J(u),\\
&J'(v_n) \to 0 \text{ in } H^{-1}(\mathbb{R}^N).
\end{aligned}$$

Proof. 1). Since $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$, we get that, as $n \to \infty$,

$$||v_n||_A^2 - ||u_n||_A^2 = (u_n - u, u_n - u)_A - ||u_n||_A^2 = ||u||_A^2 - 2(u_n, u)_A \to -||u||_A^2.$$

Therefore,

$$||v_n||_A^2 = ||u_n||_A^2 - ||u||_A^2 + o(1). \tag{5.4}$$

2). Lemma 5.5 implies

$$\int_{\mathbb{R}^N} K_*(x)(v_n^+)^p dx = \int_{\mathbb{R}^N} K_*(x)(u_n^+)^p dx - \int_{\mathbb{R}^N} K_*(x)(u^+)^p dx + o(1). \tag{5.5}$$

By (5.4), (5.5) and the assumption $J(u_n) \to c$, we get that

$$J(v_n) \to c - J(u),\ n \to \infty.$$

3). Since $J'(u_n) \to 0$ in $H^{-1}(\mathbb{R}^N)$ and $u_n \rightharpoonup u$, it is easy to verify that $J'(u) = 0$. For $h \in H^1(\mathbb{R}^N)$,

$$\begin{aligned}
\langle J'(v_n), h\rangle &= (v_n, h)_A - \int_{\mathbb{R}^N} K_*(x)(v_n^+)^{p-1} h dx\\
&= (u_n, h)_A - (u, h)_A - \int_{\mathbb{R}^N} K_*(x)(v_n^+)^{p-1} h dx.
\end{aligned}\tag{5.6}$$

By Lemma 5.4, we have

$$\begin{aligned}
&\sup_{||h||\leq 1} \left| \int_{\mathbb{R}^N} K_*(x)(v_n^+)^{p-1} h dx - \int_{\mathbb{R}^N} K_*(x)(u_n^+)^{p-1} h dx + \int_{\mathbb{R}^N} K_*(x)(u^+)^{p-1} h dx \right|\\
&\to 0,\ n \to \infty.
\end{aligned}\tag{5.7}$$

Combining (5.6) and (5.7) leads to $J'(v_n) = J'(u_n) - J'(u) + o(1)$. Then by $J'(u_n) \to 0$ in $H^{-1}(\mathbb{R}^N)$ and $J'(u) = 0$, we obtain that $J'(v_n) \to 0$ in $H^{-1}(\mathbb{R}^N)$. □

Lemma 5.7. *If* $|y_n| \to \infty$ *and as* $n \to \infty$,

$$\begin{aligned}
&u_n(\cdot + y_n) \rightharpoonup u \text{ in } H^1(\mathbb{R}^N),\\
&u_n(\cdot + y_n) \to u \ a.e. \text{ on } \mathbb{R}^N,\\
&J(u_n) \to c,\\
&J'(u_n) \to 0 \text{ in } H^{-1}(\mathbb{R}^N),
\end{aligned}$$

then there exists $\theta \in \mathbb{R}^N$ *with* $|\theta| = 1$ *such that* $J'_\theta(u) = 0$ *and* $v_n = u_n - u(\cdot - y_n)$ *is such that*

$$\begin{aligned}
&||v_n||^2 = ||u_n||^2 - ||u||^2 + o(1),\\
&J(v_n) \to c - J_\theta(u),\\
&J'(v_n) \to 0 \text{ in } H^{-1}(\mathbb{R}^N).
\end{aligned}$$

Proof. We divide the proof into several steps.

1). Since $u_n(\cdot + y_n) \rightharpoonup u$ in $H^1(\mathbb{R}^N)$, it is clear that

$$||v_n||^2 = ||v_n(\cdot + y_n)||^2 = ||u_n(\cdot + y_n)||^2 + ||u||^2 - 2(u_n(\cdot + y_n), u) = ||u_n||^2 - ||u||^2 + o(1).$$

2). For any $h \in H^1(\mathbb{R}^N)$,

$$\langle J'(u_n), h(\cdot - y_n)\rangle = (u_n, h(\cdot - y_n))_A - \int_{\mathbb{R}^N} K_*(x)(u_n^+)^{p-1} h(\cdot - y_n)dx. \tag{5.8}$$

By the definition of the inner product $(\cdot,\cdot)_A$ (see (3.11)), we have

$$\begin{aligned}
&(u_n, h(\cdot - y_n))_A\\
= \ & \int_{\mathbb{R}^N} \nabla u_n \nabla h(\cdot - y_n)dx + (\frac{b^2}{4} - b)\int_{\mathbb{R}^N} \frac{(x\cdot\nabla u_n)(x\cdot\nabla h(\cdot - y_n))}{|x|^2}dx\\
&+ \int_{\mathbb{R}^N} V_*(x) u_n h(\cdot - y_n)dx\\
= \ & \int_{\mathbb{R}^N} \nabla u_n(\cdot + y_n)\nabla h dx + a\int_{\mathbb{R}^N} u_n(\cdot + y_n)\cdot h dx\\
&+ \int_{\mathbb{R}^N} (V_*(x + y_n) - a) u_n(\cdot + y_n)\cdot h dx\\
&+ (\frac{b^2}{4} - b)\int_{\mathbb{R}^N} \Big(\frac{\frac{x}{|y_n|} + \frac{y_n}{|y_n|}}{|\frac{x}{|y_n|} + \frac{y_n}{|y_n|}|}\cdot\nabla u_n(\cdot + y_n)\Big)\Big(\frac{\frac{x}{|y_n|} + \frac{y_n}{|y_n|}}{|\frac{x}{|y_n|} + \frac{y_n}{|y_n|}|}\cdot\nabla h\Big)dx\\
:= \ & I + II + III.
\end{aligned} \tag{5.9}$$

Since $u_n(\cdot + y_n) \rightharpoonup u$ in $H^1(\mathbb{R}^N)$, we have

$$\begin{aligned}
I &= \int_{\mathbb{R}^N} \nabla u_n(\cdot + y_n)\nabla h dx + a\int_{\mathbb{R}^N} u_n(\cdot + y_n)\cdot h dx = (u_n(\cdot + y_n), h)\\
&\to \int_{\mathbb{R}^N} \nabla u \nabla h dx + a\int_{\mathbb{R}^N} u h dx, \ n \to \infty.
\end{aligned} \tag{5.10}$$

By the assumption $(\mathbf{A_2})$ and the definition of V_*, we have $\lim_{|x|\to\infty} V_*(x) = a$. This yields

$$\sup_n \int_{|x|\geq R} |V_*(x) - a|\cdot|h(x - y_n)|^2 dx \to 0, \ R \to \infty.$$

Moreover, together with (2.8) and the fact that $|y_n| \to \infty$ yields that for any fixed $R > 0$

$$\begin{aligned}
&\int_{|x|<R} |V_*(x) - a|\cdot|h(\cdot - y_n)|^2 dx\\
\leq \ & C(\int_{|x|<R} |\nabla h(\cdot - y_n)|^2 dx + \int_{|x|<R} |h(\cdot - y_n)|^2 dx) \to 0, \ n \to \infty.
\end{aligned}$$

Combining the above two limits leads to

$$\int_{\mathbb{R}^N} |V_*(x+y_n)-a|\cdot|h|^2dx \to 0,\ n\to\infty. \tag{5.11}$$

By (5.11) and the Hölder inequality, we have

$$\begin{aligned}
&|II| = |\int_{\mathbb{R}^N}(V_*(x+y_n)-a)u_n(\cdot+y_n)\cdot hdx|\\
\leq\ & (\int_{\mathbb{R}^N}|V_*(x+y_n)-a|u_n^2(\cdot+y_n)dx)^{\frac{1}{2}}(\int_{\mathbb{R}^N}|V_*(x+y_n)-a|h^2dx)^{\frac{1}{2}}\\
\leq\ & C(\int_{\mathbb{R}^N}|V_*(x+y_n)-a|h^2dx)^{\frac{1}{2}} \to 0,\ n\to\infty.
\end{aligned} \tag{5.12}$$

Since $\nabla h\in L^2(\mathbb{R}^N)$, for any $\epsilon>0$, there exists $R_\epsilon>0$ such that

$$\int_{\mathbb{R}^N\setminus\{|x|<R_\epsilon\}}|\nabla h|^2dx<\epsilon.$$

It follows that

$$\int_{\mathbb{R}^N\setminus\{|x|<R_\epsilon\}}\frac{|(\frac{x}{|y_n|}+\frac{y_n}{|y_n|})\cdot\nabla h|^2}{|\frac{x}{|y_n|}+\frac{y_n}{|y_n|}|^2}dx \leq \int_{\mathbb{R}^N\setminus\{|x|<R_\epsilon\}}|\nabla h|^2dx<\epsilon. \tag{5.13}$$

Then

$$\begin{aligned}
&|\int_{\mathbb{R}^N\setminus\{|x|<R_\epsilon\}}\Big(\frac{\frac{x}{|y_n|}+\frac{y_n}{|y_n|}}{|\frac{x}{|y_n|}+\frac{y_n}{|y_n|}|}\cdot\nabla u_n(\cdot+y_n)\Big)\Big(\frac{\frac{x}{|y_n|}+\frac{y_n}{|y_n|}}{|\frac{x}{|y_n|}+\frac{y_n}{|y_n|}|}\cdot\nabla h\Big)dx|\\
\leq\ & (\int_{\mathbb{R}^N\setminus\{|x|<R_\epsilon\}}\frac{|(\frac{x}{|y_n|}+\frac{y_n}{|y_n|})\cdot\nabla u_n(\cdot+y_n)|^2}{|\frac{x}{|y_n|}+\frac{y_n}{|y_n|}|^2}dx)^{\frac{1}{2}}\\
&\times(\int_{\mathbb{R}^N\setminus\{|x|<R_\epsilon\}}\frac{|(\frac{x}{|y_n|}+\frac{y_n}{|y_n|})\cdot\nabla h|^2}{|\frac{x}{|y_n|}+\frac{y_n}{|y_n|}|^2}dx)^{\frac{1}{2}}\\
\leq\ & (\int_{\mathbb{R}^N}|\nabla u_n|^2dx)^{1/2}(\int_{\mathbb{R}^N\setminus\{|x|<R_\epsilon\}}|\nabla h|^2dx)^{1/2}\leq C\epsilon
\end{aligned} \tag{5.14}$$

where the constant C is independent of ϵ and n. There exist a subsequence of $y_n/|y_n|$, denoted by itself for convenience, and $\theta\in\mathbb{R}^N$ with $|\theta|=1$ such that $y_n/|y_n|\to\theta$ as $n\to\infty$. Then by $|y_n|\to\infty$, we get that, as $n\to\infty$,

$$\frac{x}{|y_n|}+\frac{y_n}{|y_n|}\to\theta,\ a.e.\ \text{on } \mathbb{R}^N$$

and $\frac{x}{|y_n|}+\frac{y_n}{|y_n|}$ converges to θ uniformly for $|x|<R_\epsilon$. Therefore, there exists N_ϵ such that, when $n>N_\epsilon$,

$$\begin{aligned}
&|\int_{\{|x|<R_\epsilon\}}\Big(\frac{\frac{x}{|y_n|}+\frac{y_n}{|y_n|}}{|\frac{x}{|y_n|}+\frac{y_n}{|y_n|}|}\cdot\nabla u_n(\cdot+y_n)\Big)\Big(\frac{\frac{x}{|y_n|}+\frac{y_n}{|y_n|}}{|\frac{x}{|y_n|}+\frac{y_n}{|y_n|}|}\cdot\nabla h\Big)dx\\
&\quad-\int_{\{|x|<R_\epsilon\}}(\theta\cdot\nabla u_n(\cdot+y_n))(\theta\cdot\nabla h)dx|<\epsilon.
\end{aligned} \tag{5.15}$$

Since $u_n(\cdot+y_n)\rightharpoonup u$ in $H^1(\mathbb{R}^N)$, we have $\nabla u_n(\cdot+y_n)\rightharpoonup\nabla u$ in $L^2(\mathbb{R}^N)$. It implies that

$$\int_{\{|x|<R_\epsilon\}}(\theta\cdot\nabla u_n(\cdot+y_n))(\theta\cdot\nabla h)dx\to\int_{\{|x|<R_\epsilon\}}(\theta\cdot\nabla u)(\theta\cdot\nabla h)dx,\ n\to\infty.$$

This together with (5.14), (5.15) and

$$\int_{\mathbb{R}^N\setminus\{|x|<R_\epsilon\}} |\theta\cdot\nabla h|^2dx \le \int_{\mathbb{R}^N\setminus\{|x|<R_\epsilon\}} |\nabla h|^2dx < \epsilon,$$

yields that there exists $N'_\epsilon > 0$ such that when $n > N'_\epsilon$,

$$|\int_{\mathbb{R}^N}\Big(\frac{\frac{x}{|y_n|}+\frac{y_n}{|y_n|}}{|\frac{x}{|y_n|}+\frac{y_n}{|y_n|}|}\cdot\nabla u_n(\cdot+y_n)\Big)\Big(\frac{\frac{x}{|y_n|}+\frac{y_n}{|y_n|}}{|\frac{x}{|y_n|}+\frac{y_n}{|y_n|}|}\cdot\nabla h\Big)dx - \int_{\mathbb{R}^N}(\theta\cdot\nabla u)(\theta\cdot\nabla h)dx| < (4+C)\epsilon.$$

Thus

$$III \to (\frac{b^2}{4}-b)\int_{\mathbb{R}^N}(\theta\cdot\nabla u)(\theta\cdot\nabla h)dx,\ n\to\infty. \tag{5.16}$$

Combining (5.10), (5.12) and (5.16) leads to

$$\begin{aligned}
&(u_n, h(\cdot-y_n))_A\\
=\ &\int_{\mathbb{R}^N}\nabla u\nabla h dx + a\int_{\mathbb{R}^N} uhdx + (\frac{b^2}{4}-b)\int_{\mathbb{R}^N}(\theta\cdot\nabla u)(\theta\cdot\nabla h)dx + o(1)\\
=\ &(u,h)_\theta + o(1).
\end{aligned}\tag{5.17}$$

We obtain by the Hölder inequality and Lemma 5.2 that, as $n\to\infty$,

$$\begin{aligned}
&|\int_{\mathbb{R}^N}K_*(x+y_n)(u_n^+(\cdot+y_n))^{p-1}hdx - \mu\int_{\mathbb{R}^N}(u^+)^{p-1}hdx|\\
\le\ &C'(\int_{\mathbb{R}^N}(|u_n(\cdot+y_n)|^p+|u|^p)dx)^{\frac{p-1}{p}}(\int_{\mathbb{R}^N}|K_*(x+y_n)-\mu|^p\cdot|h|^pdx)^{\frac{1}{p}}\\
\le\ &C(\int_{\mathbb{R}^N}|K_*(x+y_n)-\mu|^p\cdot|h|^pdx)^{\frac{1}{p}}\to 0
\end{aligned}$$

where C' and C are positive constants independent of n and h. This together with (5.8) and (5.17) yields

$$\langle J'(u_n), h(\cdot-y_n)\rangle = \langle J'_\theta(u), h\rangle + o(1) \tag{5.18}$$

Then by the assumption $J'(u_n)\to 0$ in $H^{-1}(\mathbb{R}^N)$, we get $\langle J'_\theta(u),h\rangle = 0$, $\forall h\in H^1(\mathbb{R}^N)$. Therefore, $J'_\theta(u) = 0$.

3). From the definition of v_n,

$$||v_n||_A^2 = ||u_n - u(\cdot-y_n)||_A^2 = ||u_n||_A^2 + ||u(\cdot-y_n)||_A^2 - 2(u_n, u(\cdot-y_n))_A. \tag{5.19}$$

By the definition of the norm $||\cdot||_A$ (see (3.10)), we have

$$\begin{aligned}
||u(\cdot-y_n)||_A^2 =\ &\int_{\mathbb{R}^N}|\nabla u(\cdot-y_n)|^2dx + (\frac{b^2}{4}-b)\int_{\mathbb{R}^N}\frac{|x\cdot\nabla u(\cdot-y_n)|^2}{|x|^2}dx\\
&+\int_{\mathbb{R}^N}V_*(x)|u(\cdot-y_n)|^2dx\\
=\ &\int_{\mathbb{R}^N}|\nabla u|^2dx + (\frac{b^2}{4}-b)\int_{\mathbb{R}^N}\frac{|(\frac{x}{|y_n|}+\frac{y_n}{|y_n|})\cdot\nabla u|^2}{|\frac{x}{|y_n|}+\frac{y_n}{|y_n|}|^2}dx\\
&+\int_{\mathbb{R}^N}V_*(x+y_n)|u|^2dx.
\end{aligned}\tag{5.20}$$

Since $\nabla u\in L^2(\mathbb{R}^N)$ and $\frac{x}{|y_n|}+\frac{y_n}{|y_n|}\to\theta$ a.e. on $\mathbb{R}^N$, using the Lebesgue convergence theorem, we get that

$$\int_{\mathbb{R}^N}\frac{|(\frac{x}{|y_n|}+\frac{y_n}{|y_n|})\cdot\nabla u|^2}{|\frac{x}{|y_n|}+\frac{y_n}{|y_n|}|^2}dx \to \int_{\mathbb{R}^N}|\theta\cdot\nabla u|^2dx,\ n\to\infty. \tag{5.21}$$

By (5.11), (5.20) and (5.21), we get that

$$\begin{aligned} ||u(\cdot - y_n)||_A^2 &= \int_{\mathbb{R}^N} |\nabla u|^2 dx + (\frac{b^2}{4} - b)\int_{\mathbb{R}^N} |\theta \cdot \nabla u|^2 dx + a\int_{\mathbb{R}^N} |u|^2 dx + o(1) \\ &= ||u||_\theta^2 + o(1). \end{aligned} \tag{5.22}$$

Combining (5.19), (5.22) and (5.17) leads to

$$||v_n||_A^2 = ||u_n||_A^2 - ||u||_\theta^2 + o(1). \tag{5.23}$$

Note that

$$\begin{aligned} & \int_{\mathbb{R}^N} K_*(x)(v_n^+)^p dx \\ = & \int_{\mathbb{R}^N} K_*(x+y_n)((u_n(\cdot + y_n) - u)^+)^p dx \\ = & \int_{\mathbb{R}^N} ((K_*^{\frac{1}{p}}(x+y_n)u_n(\cdot + y_n) - K_*^{\frac{1}{p}}(x+y_n)u)^+)^p dx. \end{aligned} \tag{5.24}$$

We obtain from Lemma 5.5 that

$$\begin{aligned} & \int_{\mathbb{R}^N} ((K_*^{\frac{1}{p}}(x+y_n)u_n(\cdot + y_n) - K_*^{\frac{1}{p}}(x+y_n)u)^+)^p dx \\ = & \int_{\mathbb{R}^N} ((K_*^{\frac{1}{p}}(x+y_n)u_n^+(\cdot + y_n))^p dx - \int_{\mathbb{R}^N} ((K_*^{\frac{1}{p}}(x+y_n)u^+)^p dx + o(1) \\ = & \int_{\mathbb{R}^N} K_*(x)(u_n^+)^p dx - \int_{\mathbb{R}^N} K_*(x+y_n)(u^+)^p dx + o(1). \end{aligned} \tag{5.25}$$

By Lemma 5.2,

$$\int_{\mathbb{R}^N} K_*(x+y_n)(u^+)^p dx = \mu \int_{\mathbb{R}^N} (u^+)^p dx + o(1). \tag{5.26}$$

Combining $(5.24) - (5.26)$ yields

$$\int_{\mathbb{R}^N} K_*(x)(v_n^+)^p dx = \int_{\mathbb{R}^N} K_*(x)(u_n^+)^p dx - \mu \int_{\mathbb{R}^N} (u^+)^p dx + o(1). \tag{5.27}$$

Combining (5.23), (5.27) and the assumption $J(u_n) \to c$ leads to

$$J(v_n) = J(u_n) - J_\theta(u) + o(1) = c - J_\theta(u) + o(1).$$

4). For $h \in H^1(\mathbb{R}^N)$,

$$\begin{aligned} \langle J'(v_n), h\rangle &= (v_n, h)_A - \int_{\mathbb{R}^N} K_*(x)(v_n^+)^{p-1} h dx \\ &= (u_n, h)_A - (u(\cdot - y_n), h)_A - \int_{\mathbb{R}^N} K_*(x)(v_n^+)^{p-1} h dx. \end{aligned} \tag{5.28}$$

We shall give the limits for $(u(\cdot - y_n), h)_A$ and $\int_{\mathbb{R}^N} K_*(x)(v_n^+)^{p-1} h dx$ as $n \to \infty$.

First, as (5.9), we have

$$\begin{aligned} & (u(\cdot - y_n), h)_A \\ = & \int_{\mathbb{R}^N} \nabla u \nabla h(\cdot + y_n) dx + a\int_{\mathbb{R}^N} u \cdot h(\cdot + y_n) dx \\ & + \int_{\mathbb{R}^N} (V_*(x+y_n) - a) u \cdot h(\cdot + y_n) dx \\ & + (\frac{b^2}{4} - b)\int_{\mathbb{R}^N} \Big(\frac{\frac{x}{|y_n|} + \frac{y_n}{|y_n|}}{|\frac{x}{|y_n|} + \frac{y_n}{|y_n|}|} \cdot \nabla u\Big)\Big(\frac{\frac{x}{|y_n|} + \frac{y_n}{|y_n|}}{|\frac{x}{|y_n|} + \frac{y_n}{|y_n|}|} \cdot \nabla h(\cdot + y_n)\Big) dx. \end{aligned}$$

By the Hölder inequality and (5.11), we get that if $||h||\leq 1$, then

$$
\begin{aligned}
& |\int_{\mathbb{R}^N}(V_*(x+y_n)-a)u\cdot h(\cdot+y_n)dx| \\
\leq\ & (\int_{\mathbb{R}^N}|V_*(x+y_n)-a|\cdot u^2dx)^{1/2}(\int_{\mathbb{R}^N}|V_*(x)-a|h^2dx)^{1/2} \\
\leq\ & C(\int_{\mathbb{R}^N}|V_*(x+y_n)-a|\cdot u^2dx)^{1/2}\to 0,\ n\to\infty.
\end{aligned}
$$

Thus, as $n\to\infty$,

$$\int_{\mathbb{R}^N}(V_*(x+y_n)-a)u\cdot h(\cdot+y_n)dx=o(1)$$

holds uniformly for $||h||\leq 1$. Moreover, a similar argument as the proof of (5.16) yields that, as $n\to\infty$,

$$
\begin{aligned}
& \int_{\mathbb{R}^N}\Big(\frac{\frac{x}{|y_n|}+\frac{y_n}{|y_n|}}{|\frac{x}{|y_n|}+\frac{y_n}{|y_n|}|}\cdot\nabla u\Big)\Big(\frac{\frac{x}{|y_n|}+\frac{y_n}{|y_n|}}{|\frac{x}{|y_n|}+\frac{y_n}{|y_n|}|}\cdot\nabla h(\cdot+y_n)\Big)dx \\
=\ & \int_{\mathbb{R}^N}(\theta\cdot\nabla u)(\theta\cdot\nabla h(\cdot+y_n))dx+o(1)
\end{aligned}
$$

holds uniformly for $||h||\leq 1$. Therefore, as $n\to\infty$,

$$(u(\cdot-y_n),h)_A=(u,h(\cdot+y_n))_\theta+o(1) \tag{5.29}$$

holds uniformly for $||h||\leq 1$.

Second, from $u_n(\cdot+y_n)\rightharpoonup u$ in $H^1(\mathbb{R}^N)$ and Lemma 5.4, we deduce that, as $n\to\infty$,

$$
\begin{aligned}
& |\int_{\mathbb{R}^N}K_*(x+y_n)((u_n(\cdot+y_n)-u)^+)^{p-1}h(\cdot+y_n)dx \\
& -\int_{\mathbb{R}^N}K_*(x+y_n)((u_n^+(\cdot+y_n))^{p-1}h(\cdot+y_n)dx \\
& +\int_{\mathbb{R}^N}K_*(x+y_n)(u^+)^{p-1}h(\cdot+y_n)dx|\to 0
\end{aligned} \tag{5.30}
$$

holds uniformly for $||h||\leq 1$. By the Hölder inequality, (3.14) and Lemma 5.2, we get that, if $||h||\leq 1$, then

$$
\begin{aligned}
& |\int_{|x|>R}K_*(x+y_n)(u^+)^{p-1}h(\cdot+y_n)dx| \\
\leq\ & (\int_{|x|>R}K_*(x+y_n)(u^+)^pdx)^{\frac{p-1}{p}}(\int_{|x|>R}K_*(x+y_n)|h|^pdx)^{1/p} \\
\leq\ & C(\int_{|x|>R}K_*(x+y_n)(u^+)^pdx)^{\frac{p-1}{p}}\to 0,\ R\to\infty
\end{aligned} \tag{5.31}
$$

By Lemma 5.2, we get that, for every $R>0$, as $n\to\infty$,

$$
\begin{aligned}
& \sup_{||h||\leq 1}|\int_{|x|\leq R}(K_*(x+y_n)-\mu)(u^+)^{p-1}h(\cdot+y_n)dx| \\
\leq\ & \sup_{||h||\leq 1}(\int_{|x|\leq R}|K_*(x+y_n)-\mu|(u^+)^pdx)^{\frac{p-1}{p}}(\int_{\mathbb{R}^N}|K_*(x)-\mu|\cdot|h|^pdx)^{1/p} \\
\leq\ & C(\int_{|x|\leq R}|K_*(x+y_n)-\mu|(u^+)^pdx)^{\frac{p-1}{p}}\to 0.
\end{aligned} \tag{5.32}
$$

Combining (5.31) and (5.32) yields that

$$\int_{\mathbb{R}^N} K_*(x+y_n)(u^+)^{p-1}h(\cdot+y_n)dx - \mu\int_{\mathbb{R}^N}(u^+)^{p-1}h(\cdot+y_n)dx \to 0 \tag{5.33}$$

holds uniformly for $||h|| \le 1$. Then by (5.30), (5.33) and

$$\int_{\mathbb{R}^N} K_*(x)(v_n^+)^{p-1}hdx = \int_{\mathbb{R}^N} K_*(x+y_n)((u_n(\cdot+y_n)-u)^+)^{p-1}hdx,$$

we get that, as $n\to\infty$,

$$\begin{aligned}&|\int_{\mathbb{R}^N} K_*(x)(v_n^+)^{p-1}hdx - \int_{\mathbb{R}^N} K_*(x)(u_n^+)^{p-1}hdx + \mu\int_{\mathbb{R}^N}(u^+)^{p-1}h(\cdot+y_n)dx| \\ &\to 0\end{aligned} \tag{5.34}$$

holds uniformly for $||h|| \le 1$.

Finally, combining (5.28), (5.29) and (5.34) leads to

$$\langle J'(v_n), h\rangle - \langle J'(u_n), h\rangle + \langle J'_\theta(u), h(\cdot+y_n)\rangle \to 0$$

holds uniformly for $||h|| \le 1$. This together with the fact that $J'_\theta(u) = 0$ and $J'(u_n) \to 0$ in $H^{-1}(\mathbb{R}^N)$ yields $J'(v_n) \to 0$ in $H^{-1}(\mathbb{R}^N)$. □

Proof of Theorem 5.1. We divide the proof into two steps.

1). For n big enough, we have

$$c+1+||u_n|| \ge J(u_n) - p^{-1}\langle J'(u_n), u_n\rangle = (\frac{1}{2}-\frac{1}{p})||u_n||_A^2. \tag{5.35}$$

As mentioned in section 3, the norm $||\cdot||_A$ is equivalent to the norm $||\cdot||$. Therefore, there exists a constant $C>0$ such that $||u||_A \ge C||u||$, $\forall u \in H^1(\mathbb{R}^N)$. Then by (5.35), there exists a constant $C'>0$ such that for n big enough,

$$c+1+||u_n|| \ge C'||u_n||^2$$

It follows that $||u_n||$ is bounded.

2). Assume that $u_n \rightharpoonup u_0$ in $H^1(\mathbb{R}^N)$ and $u_n \to u_0$ a.e. on $\mathbb{R}^N$. By Lemma 5.6, $J'(u_0) = 0$ and $u_n^1 = u_n - u_0$ is such that

$$\begin{aligned}&||u_n^1||_A^2 = ||u_n||_A^2 - ||u_0||_A^2 + o(1),\\ &J(u_n^1) \to c - J(u),\\ &J'(u_n^1) \to 0 \text{ in } H^{-1}(\mathbb{R}^N).\end{aligned} \tag{5.36}$$

Let us define

$$\delta := \overline{\lim}_{n\to\infty} \sup_{y\in\mathbb{R}^N}\int_{|x-y|\le 1}|u_n^1|^2dx.$$

If $\delta = 0$, Lemma 5.3 implies that $K_*^{1/p}u_n^1 \to 0$ in $L^p(\mathbb{R}^N)$. Since $J'(u_n^1) \to 0$ in $H^1(\mathbb{R}^N)$, it follows that

$$||u_n^1||_A^2 = \langle J'(u_n^1), u_n^1\rangle + \int_{\mathbb{R}^N} K_*(x)((u_n^1)^+)^p dx \to 0$$

and the proof is complete. If $\delta > 0$, we may assume the existence of $\{y_n^1\} \subset \mathbb{R}^N$ such that

$$\int_{|x-y_n^1|\le 1}|u_n^1|^2dx > \delta/2.$$

Let us define $v_n^1 := u_n^1(\cdot + y_n^1)$. We may assume that $v_n^1 \rightharpoonup u_1$ in $H^1(\mathbb{R}^N)$ and $v_n^1 \to u_1$ a.e. on $\mathbb{R}^N$. Since

$$\int_{|x|\leq 1} |v_n^1|^2 dx > \delta/2$$

it follows from the Rellich Theorem that

$$\int_{|x|\leq 1} |u^1|^2 dx \geq \delta/2$$

and $u_1 \neq 0$. But $u_n^1 \rightharpoonup 0$ in $H^1(\mathbb{R}^N)$, so that $\{|y_n^1|\}$ is unbounded. We may assume that $|y_n^1| \to \infty$. Finally, by (5.36) and Lemma 5.7, there exists $\theta_1 \in \mathbb{R}^N$ with $|\theta_1| = 1$ such that $J'_{\theta_1}(u_1) = 0$ and $u_n^2 := u_n^1 - u_1(\cdot - y_n^1)$ satisfies

$$\begin{aligned}
&||u_n^2||^2 = ||u_n^1||^2 - ||u_1||^2 + o(1),\\
&J(u_n^2) \to c - J_{\theta_1}(u_1),\\
&J'(u_n^2) \to 0 \text{ in } H^{-1}(\mathbb{R}^N).
\end{aligned}$$

Moreover, Lemma 4.3 implies that

$$J_{\theta_1}(u_1) \geq (\frac{1}{2} - \frac{1}{p})\mu^{-\frac{2}{p-2}} S^{\frac{p}{p-2}}.$$

Iterating the above procedure we construct sequences $\{\theta_l\}$, $\{u_l\}$ and $\{y_n^l\}$. Since for every l, $J_{\theta_l}(u_l) \geq (\frac{1}{2} - \frac{1}{p})\mu^{-\frac{2}{p-2}} S^{\frac{p}{p-2}}$, the iteration must terminate at some finite index k. This finishes the proof of this theorem. □

The following corollary is a direct consequence of Theorem 5.1 and Lemma 4.3. It implies that the functional J satisfies $(PS)_c$ condition if $c < (\frac{1}{2} - \frac{1}{p})\mu^{-\frac{2}{p-2}} S^{\frac{p}{p-2}}$.

Corollary 5.8. *Under the assumptions* $(\mathbf{A_1})$ *and* $(\mathbf{A_2})$*, any sequence* $\{u_n\} \subset H^1(\mathbb{R}^N)$ *such that*

$$J(u_n) \to c < (\frac{1}{2} - \frac{1}{p})\mu^{-\frac{2}{p-2}} S^{\frac{p}{p-2}},\ J'(u_n) \to 0 \textit{ in } H^{-1}(\mathbb{R}^N)$$

contains a convergent subsequence.

6 Proof of Theorem 1.1.

Recall that the critical points of J are nonnegative solutions of (2.9). By Corollary 2.2, to prove Eq.(1.1) has a positive solution, it suffices to prove that J has a nontrivial critical point. And by Corollary 5.8, it suffices to apply the classical mountain pass theorem (see, e.g., [14, Theorem 1.15]) to J with the mountain pass value $c < (\frac{1}{2} - \frac{1}{p})\mu^{-\frac{2}{p-2}} S^{\frac{p}{p-2}}$.

By the assumption (1.10) and Lemma 4.2, there exists a nonnegative $u_0 \in H^1(\mathbb{R}^N) \setminus \{0\}$ such that

$$\frac{||u_0||_A^2}{(\int_{\mathbb{R}^N} K_*(x) u_0^p dx)^{2/p}} < (1 - b/2)^{\frac{p-2}{p}} \mu^{-\frac{2}{p}} S_p = \mu^{-\frac{2}{p}} S.$$

We obtain

$$\begin{aligned}
0 < \max_{t\geq 0} J(tu_0) &= \max_{t\geq 0} \Big(\frac{t^2}{2} ||u_0||_A^2 - \frac{t^p}{p} \int_{\mathbb{R}^N} K_*(x)(u_0^+)^p dx \Big)\\
&= (\frac{1}{2} - \frac{1}{p}) \Big(||u_0||_A^2 \Big/ (\int_{\mathbb{R}^N} K_*(x) u_0^p dx)^{2/p} \Big)^{\frac{p}{p-2}}\\
&< (\frac{1}{2} - \frac{1}{p})\mu^{-\frac{2}{p-2}} S^{\frac{p}{p-2}}. \qquad (6.1)
\end{aligned}$$

By (3.14),

$$J(u) \geq \frac{1}{2}||u||_A^2 - \frac{C^p}{p}||u||_A^p.$$

Therefore, there exists $r > 0$ such that

$$b := \inf_{||u||_A = r} J(u) > 0 = J(0).$$

Moreover, there exists $t_0 > 0$ such that $||t_0 u_0||_A > r$ and $J(t_0 u_0) < 0$. It follows from (6.1) that

$$\max_{t\in[0,1]} J(tt_0u_0) < (\frac{1}{2} - \frac{1}{p})\mu^{-\frac{2}{p-2}} S^{\frac{p}{p-2}}.$$

By Corollary 5.8 and the mountain pass theorem (see [14, Theorem 1.15]), J has a critical value c such that $b \leq c < (\frac{1}{2} - \frac{1}{p})\mu^{-\frac{2}{p-2}} S^{\frac{p}{p-2}}$ and Eq.(2.9) has a positive solution $v \in H^1(\mathbb{R}^N)$. Then by Theorem 2.2, the function u defined by (2.1) is a positive solution of (1.1). To complete the proof, it suffices to prove that $u \in E$. Using the divergence theorem, Lemma 2.1 and (2.12), we get that

$$\begin{aligned}
& \int_{\mathbb{R}^N} |\nabla u|^2 dx \\
= & -\int_{\mathbb{R}^N} u \triangle u dx \\
= & -\int_{\mathbb{R}^N} u \cdot |y|^{-\frac{b(N+2)}{2(2-b)}} \Big(\sum_{i,j=1}^N \frac{\partial}{\partial y_j}\Big(A_{ij}(y)\frac{\partial v}{\partial y_i}\Big) - \frac{C_b}{|y|^2} v\Big) dx \\
= & -\int_{\mathbb{R}^N} |x|^{-\frac{b}{4}(N-2)} v(|x|^{-\frac{b_*}{2}} x) \cdot |y|^{-\frac{b(N+2)}{2(2-b)}} \Big(\sum_{i,j=1}^N \frac{\partial}{\partial y_j}\Big(A_{ij}(y)\frac{\partial v}{\partial y_i}\Big) - \frac{C_b}{|y|^2} v\Big) dx \\
= & -\int_{\mathbb{R}^N} v \cdot \Big(\sum_{i,j=1}^N \frac{\partial}{\partial y_j}\Big(A_{ij}(y)\frac{\partial v}{\partial y_i}\Big) - \frac{C_b}{|y|^2} v\Big) dy \\
= & \int_{\mathbb{R}^N} \Big(|\nabla v|^2 + \frac{|x \cdot \nabla v|^2}{|x|^2} + \frac{C_b}{|x|^2} v^2 \Big) dy.
\end{aligned}$$

Moreover, by Lemma 2.1 and (2.12), we get that

$$\int_{\mathbb{R}^N} V(x) u^2 dx = \int_{\mathbb{R}^N} V(x)|x|^{-\frac{b}{2}(N-2)} v^2(|x|^{-\frac{b}{2}} x) dx = \int_{\mathbb{R}^N} V_*(y) v^2 dy. \tag{6.2}$$

Therefore, $||u||_E^2 = ||v||_A^2 < \infty$. □

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